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Strongly self-dual graphs

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ABSTRACT

We present a class of graphs whose adjacency matrices are nonsingular with integral inverses, denoted *h*-graphs. If the *h*-graphs G and H with adjacency matrices $M(G)$ and $M(H)$ satisfy $M(G)^{-1} = SM(H)S$, where S is a signature matrix, we refer to H as the dual of G. The dual is a type of graph inverse. If the *h*-graph G is isomorphic to its dual via a particular isomorphism, we refer to G as strongly self-dual. We investigate the structural and spectral properties of strongly self-dual graphs, with a particular emphasis on identifying when such a graph has 1 as an eigenvalue.

1. Preliminaries

We consider graphs that have no loops or multiple edges. The adjacency matrix of the graph G is the $(0, 1)$-matrix $M(G)$ that satisfies $m_{ij} = 1$ if and only if vertices i and j are adjacent. Graphs G and H are *isomorphic* ($G \cong H$) if one can be obtained by relabeling the vertices of the other. In an undirected graph, xy denotes the edge with endpoints x and y. In a directed graph (digraph), $x \rightarrow y$ denotes the arc with initial vertex x and terminal vertex y; as well, $x \rightsquigarrow y$ denotes a directed path from x to y. A digraph is *acyclic* if it contains no directed cycles.

The idea of identifying the inverse of a graph has received a great deal of research interest. Let G be a graph with adjacency matrix $M = M(G)$. A *graph-inverse* of G is a second graph H that can be associated, in some manner, with M^{-1}. For example, in [4], the *signed inverse* is defined as follows: if G and H are graphs such that $M(G)^{-1}$ is a $(0, \pm 1)$-matrix and $M(H)$ is obtained by replacing each -1 in $M(G)^{-1}$ with 1 we refer to H as the signed inverse of G. When M is singular, it is still possible to define a

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graph-inverse of G. In [2], graphs G and H are said to be inverses of each other if whenever λ is a nonzero eigenvalue of one graph, $1/\lambda$ is an eigenvalue of the other. As well, the *pseudo-inverse* of a graph, defined in [5], is an interesting example of a graph-inverse that is defined in purely graph-theoretic terms.

Graph inverses appear in connection to quantum chemistry. In Hückel theory, a graph known as the *Hückel graph* is used to model molecular orbital energies. It has been shown that many families of Hückel graphs are bipartite and possess a unique perfect matching. (See [7,11] for further information on Hückel theory.) We refer to a graph that is bipartite and possesses a unique perfect matching as an *h-graph*. (Although h-graphs are named after Hückel graphs, neither family of graphs is contained in the other.) The class of h-graphs has been of great interest in research into the inverse of a graph; see [6] and Theorems 2.8 and 2.9.

1.1. Partial orders and digraphs

A *poset* is a set X together with a partial order $\preceq$ on X. An *interval* in a poset is a subset of the form

$$[x, y] = \{z : x \preceq z \preceq y\}\,.$$

A *covering pair* (or, a cover) in a poset is a pair $x \prec y$ where no element z satisfies $x \prec z \prec y$. A poset is uniquely determined by its covers. Alternately, one could define a cover as a pair $x \neq y$ such that

$$[x, y] = \{x, y\}\,.$$

The *Hasse diagram* of a poset is the digraph on X that has $x \to y$ if and only if $x \prec y$ is a covering pair.

A digraph D induces a partial order on its vertices: $\alpha \prec \beta$ if and only if there is a directed path $\alpha \rightsquigarrow \beta$. If $\alpha \prec \beta$ is a cover in this partial order, there must be an arc $\alpha \to \beta$, which we refer to as a covering arc. We define the digraph $\Gamma(D)$ to be the subgraph of D that includes only the covering arcs. We note that $\Gamma(D)$ is the Hasse diagram of the partial order induced by D; thus there is a directed path $\alpha \rightsquigarrow \beta$ in D if and only if there is a directed path $\alpha \rightsquigarrow \beta$ in $\Gamma(D)$.

Let D be an acyclic digraph on an indexed vertex set $\{\alpha_i\}$. We define the *directed interval* $D[i, j]$ to be the induced subgraph of D on the vertex set $[\alpha_i, \alpha_j]$. We say that D is a directed interval if there are vertices α_i and α_j such that $D = D[i, j]$. When this holds, we refer to α_i and α_j as the *extremal vertices* of the directed interval D.

1.2. h-Graphs and their associated structures

A *matching* in a graph is collection of nonincident edges (i.e. no two have a common endpoint). A *perfect matching* is a matching that includes every vertex as an endpoint. It is clear that if G has a perfect matching then G contains $2m$ vertices where m is the number of edges in the perfect matching. A graph has a *unique perfect matching* if there is exactly one such collection of edges.

Let the graph G possess a unique perfect matching. We refer to the edges in the perfect matching as *matched* edges. An *alternating path* in G is a path that has its first, last and every second edge matched.

An *h-graph* is a bipartite graph that has a unique perfect matching. Proposition 1.1 is taken from [8].

Proposition 1.1. *Let G be a graph; then G is an h-graph if and only if the adjacency matrix of G can be expressed as*

$$M(G) = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix},$$

where B is a lower-triangular, square $(0, 1)$*-matrix with every diagonal entry equal to* 1.

Let G be an h-graph with adjacency matrix as above. We label $B_G = B$ and $N_G = B_G - I$. The matrix N_G is a strictly lower-triangular $(0, 1)$-matrix; thus, it is the adjacency matrix of an acyclic digraph, which we label D_G.

The adjacency matrix of a graph has multiple expressions; thus, there are multiple possibilities for D_G. However, Proposition 1.2 shows that different representations of D_G have essentially the same structure. We will use the vertex labeling in Proposition 1.3 as the canonical vertex labeling for an h-graph and its associated digraph. See [9] for a thorough exposition of Propositions 1.2 and 1.3.

Proposition 1.2. *Let G be a connected h-graph and let*

$$M_1 = \begin{bmatrix} 0 & B_1 \\ B_1^T & 0 \end{bmatrix} \quad \text{and} \quad M_2 = \begin{bmatrix} 0 & B_2 \\ B_2^T & 0 \end{bmatrix}$$

be representations of the adjacency matrix of G as in Proposition 1.1. Let the digraphs D_1 and D_2 have adjacency matrices $N_1 = B_1 - I$ and $N_2 = B_2 - I$, respectively. Then $D_2 \cong D_1$ or $D_2 \cong D_1^R$, where D_1^R is obtained by reversing the orientation of every arc in D_1.

A weakly connected component in a digraph is an induced subgraph that corresponds to a connected component in the underlying undirected graph. In light of Proposition 1.2, we will say that digraphs D_1 and D_2 are equivalent ($D_1 \cong D_2$) if one can be obtained from the other by relabeling the vertices and/or reversing the orientation of every arc in one or more weakly connected components. Under this equivalence relation, the digraph associated with an h-graph is uniquely determined. Note that if D^R is obtained by reversing the orientations of the arcs in D, then the adjacency matrix of D^R is N^T where N is the adjacency matrix of D.

Proposition 1.3. *Let G be an h-graph on $2m$ vertices with adjacency matrix*

$$M = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix},$$

as in Proposition 1.1. For $1 \leqslant i \leqslant m$, let u_i be the vertex indexed by the ith row/column and let v_i be the vertex indexed by the $(i+m)$th row/column of M. Let $D = D_G$ have adjacency matrix $N = B - I$ and let vertex α_i (in D) be indexed by the ith row and column of N. Then,

1. *the unique perfect matching of G consists of the edges $e_i = u_i v_i$;*
2. *the arc $\alpha_i \to \alpha_j$ is present in D if and only if the edge $u_i v_j$ is present in G; and*
3. *there is a directed path $\alpha_i \rightsquigarrow \alpha_j$ in D if and only if there is an alternating path with endpoints v_i and u_j in G.*

2. Strongly self-dual graphs

2.1. The dual of an h-graph

A *signature matrix* is a diagonal matrix with every diagonal entry equal to ± 1. If S is a signature matrix, we use the shorthand $s_i = s_{ii}$. Note that for any signature matrix S we have $S = S^T = S^{-1}$. We say that the matrices X and Y are *signature-similar* if there is a signature matrix S such that $SXS = Y$.

Let G be an h-graph on $2m$ vertices with adjacency matrix M. By Proposition 1.1, the determinant of M is

$$\det(M) = (-1)^m \det(B_G)^2 = (-1)^m .$$

Thus the adjacency matrix of G is nonsingular with integral inverse.

Definition 2.1. Let G be an h-graph and let M be the adjacency matrix of G. If there is a signature matrix S such that $SM^{-1}S$ is a $(0, 1)$-matrix, then this matrix is itself the adjacency matrix of another h-graph. We label this graph G^+ and refer to it as the *dual of* G.

The dual of an h-graph G is a type of graph-inverse: its spectrum is the inverse spectrum of G and its adjacency matrix can be obtained by replacing every nonzero entry in M^{-1} with a 1. We note that not every h-graph possesses a dual.

Note that for any matrix X and any signature matrix S, we have $SXS = (-S)X(-S)$. Thus, to simplify matters, we will usually assume that in a signature matrix S we have $s_1 = -1$.

Theorems 2.2–2.6 and their proofs are found in [9]. We will use these to prove Theorem 3.2, which is a parallel version of Theorem 2.6 concerning strongly self-dual graphs.

Theorem 2.2. *Let G be an h-graph and let $B = B_G$. Then, G possesses a dual if and only if there is a signature matrix S such that $SB^{-1}S$ is a $(0, 1)$-matrix. When this holds, the adjacency matrix of G^+ can be expressed as*

$$M^+ = \begin{bmatrix} 0 & B^+ \\ B^{+T} & 0 \end{bmatrix}$$

where $B^+ = SB^{-1}S$ and $B^{+T} = (B^+)^T$.

We will take Theorem 2.2 to be the canonical expression of the adjacency matrix of the dual of an h-graph. Note that the dual of the h-graph G is unique (when it exists) and that, in fact, $(G^+)^+ = G$. To see this, we simply note that

$$B_{G^+} = B^+ = SB^{-1}S;$$

thus,

$$S\left(B^+\right)^{-1}S = S\,(SBS)\,S = B$$

implies that $(B^+)^+ = B$.

Recall that $\Gamma(D)$ is the Hasse diagram of the digraph D. For G an h-graph, we define $\Gamma_G = \Gamma(D_G)$.

Theorem 2.3. *Let G be an h-graph that possesses a dual, let $B = B_G$ and let $\Gamma = \Gamma_G$. Then, Γ is bipartite. Moreover, for any signature matrix S, $SB^{-1}S$ is a $(0, 1)$-matrix if and only if $s_i = -s_j$ whenever $\alpha_i \to \alpha_j$ is a covering arc.*

Note that Theorem 2.2 does not claim that having $s_i = -s_j$ along any covering arc is a sufficient condition for possessing a dual. It simply says the following: let $\mathcal{S}$ be the collection of signature matrices that conjugate B_G^{-1} to a $(0, 1)$-matrix and let $\mathcal{T}$ be those that have $s_i = -s_j$ along covering arcs in D_G; if $\mathcal{S} \neq \emptyset$ then $\mathcal{S} = \mathcal{T}$.

Corollary 2.4. *Let G be a connected h-graph that possesses a dual and let $B = B_G$. Then there are exactly two signature matrices S such that $SB^{-1}S = B^+$. One of these, which we refer to as S_1, can be obtained by letting $s_1 = -1$, and setting $s_i = -s_j$ whenever $i \prec j$ is a cover in D_G. The other is then $S_2 = -S_1$.*

Let G be an h-graph and let $D = D_G$. Let $D[i, j]$ be a directed interval in D and let N_1 be the adjacency matrix of $D[i, j]$. Then, N_1 is a principal submatrix of N_G and so

$$M_1 = \begin{bmatrix} 0 & I + N_1 \\ I + N_1^T & 0 \end{bmatrix}$$

is a principal submatrix of the adjacency matrix of G. Thus, M_1 is the adjacency matrix of an induced subgraph of G, which we refer to as the *undirected interval* $G[i,j]$. Via Proposition 1.3 and the definition of a directed interval, x is present in $G[i,j]$ if and only if there is an alternating path in G with endpoints v_i and u_j that contains x.

Theorem 2.5. *Let G be an h-graph and let G^+ be the dual of G. If $G[i,j]$ is an undirected interval in G, then $G[i,j]$ possesses a dual. Moreover, the dual of $G[i,j]$ is the corresponding undirected interval $G^+[i,j]$ in G^+.*

Theorem 2.6. *Let G be an h-graph. Then G possesses a dual, G^+, if and only if both of the following conditions hold:*

1. *Each nonempty undirected interval $G[i,j]$ in G possesses a dual.*
2. *The digraph Γ_G is bipartite.*

Theorem 2.6 is a central result in the study of the dual of an h-graph. In [9], we use it to characterize dual pairs of h-graphs, G and G^+, such that both graphs are either unicyclic or acyclic.

2.2. Self-dual graphs

Let G be a connected h-graph; if G^+ is well-defined and is isomorphic to G, we say that G is *self-dual*. Now, if G is any h-graph that has a dual, we can form the disjoint union of G and its dual, $H = G + G^+$. Such an H is trivially self-dual; however this classification does not illuminate any interesting structures in G. So, in any discussion of self-dual h-graphs, we will only consider connected h-graphs.

Theorem 2.7. *Let G be a connected h-graph and let $B = B_G$. Then, G is self-dual if and only if there is a permutation matrix P and a signature matrix S such that $PBP^T = SB^{-1}S$ or $PB^TP^T = SB^{-1}S$.*

Proof. Suppose that G is a connected and self-dual h-graph on $2m$ vertices. Let the adjacency matrix of G be as in Proposition 1.1; let signature matrix S be such that $SB^{-1}S = B^+$. By Theorem 2.2, the adjacency matrix of G^+ can be expressed as

$$M^+ = \begin{bmatrix} 0 & B^+ \\ B^{+T} & 0 \end{bmatrix}.$$

Let Q be a permutation matrix such that $QMQ^+ = M^+$. Since G is connected and bipartite, it has a unique bipartition and so Q must have one of two forms:

$$Q = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & P_1 \\ P_2 & 0 \end{bmatrix}.$$

Since G has a unique perfect matching, conjugation by Q must fix the submatrix

$$\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$$

of M. Thus, in either case, $P_1 = P_2$, and the statement holds.

The converse is trivial. $\square$

2.3. Strongly self-dual graphs

We introduce a refinement of the concept of self-duality. Let G be an h-graph and let $B = B_G$. We say that G is *strongly self-dual* if there is a signature matrix S such that $SBS = B^{-1}$. That is, G is strongly self-dual if $B^{+} = B$.

Theorem 2.7 shows that there are multiple ways in which a graph can be self-dual, investigations of their structure seem to require a case-by-case approach (see, for example, [9]). The simpler definition of strongly self-dual graphs allows us to proceed in a more direct manner. Moreover, strongly self-dual graphs are a generalization of a very well-known family of invertible graphs, namely nonsingular trees.

Let G be a graph on n vertices $\{x_i\}$. The *corona* of G, denoted $C(G)$, is the graph obtained by adding n new vertices $\{x_i'\}$ together with the n edges $\{x_i'x_i\}$ to G.

Theorems 2.8 and 2.9 are very thorough characterizations of invertible trees; they are the result of multiple pieces of research into duals of graphs, see [1,3,6,8]. Our goal is to characterize larger families of strongly self-dual graphs.

Theorem 2.8. *Let G be a bipartite graph. Then, $C(G)$ is a strongly self-dual h-graph.*

Theorem 2.9. *Let T be a tree. Then, T possesses a dual if and only if it is an h-graph; moreover, when T^{+} exists, T is a subgraph of T^{+}. Further, T is self-dual if and only if T is the corona of another tree; thus, every self-dual tree is, in fact, strongly self-dual.*

3. Structure of strongly self-dual graphs

Theorem 3.1. *Let G be a strongly self-dual h-graph and let $G[i,j]$ be an undirected interval in G. Then, $G[i,j]$ is a strongly self-dual h-graph.*

Proof. Let $D = D_G$, $B = B_G$ and $N = N_G$. Let $i \neq j$ be such that $\alpha_i \prec \alpha_j$ in D. Let N_1 be the principal submatrix of N corresponding to $D[i,j]$, let N_0 be the principal submatrix corresponding to those α with $\alpha_i \prec \alpha$ and $\alpha \not\prec \alpha_j$ and N_2 be the principal submatrix corresponding to the remaining vertices. Thus, we can express

$$N = \begin{bmatrix} N_0 & 0 & 0 \\ * & N_1 & 0 \\ * & * & N_2 \end{bmatrix}.$$

Let S be such that $SBS = B^{-1}$ and let

$$S = \begin{bmatrix} S_0 & 0 & 0 \\ 0 & S_1 & 0 \\ 0 & 0 & S_2 \end{bmatrix}$$

be the corresponding decomposition of S. Since $S(I+N)S = (I+N)^{-1}$ and the inverse of a block lower-triangular matrix has the inverses of the diagonal elements on its diagonal, we have $S_1(I+N_1)S_1 = (I+N_1)^{-1}$. Then we note that $B_{G[i,j]} = I + N_1$. $\square$

Theorem 3.2. *Let G be an h-graph. Then, G is strongly self-dual if and only if both of the following conditions hold:*

1. *Every undirected interval $G[i,j]$ in G is strongly self-dual.*
2. *The Hasse Diagram Γ_G is bipartite.*

Proof. If G is strongly self-dual, then conditions 1 and 2 are simply Theorems 2.3 and 3.1.

Suppose that G is an h-graph that satisfies conditions 1 and 2. By Theorem 2.6, G^+ exists. Let $D = D_G$, $D^+ = D_{G^+}$ and $B = B_G$. Condition 1 and Theorem 2.5 together imply that for every $i \prec j$ we have

$$G[i,j] = (G[i,j])^+ = G^+[i,j].$$

And so we also have $D[i,j] = D^+[i,j]$ for all i and j. This implies that there is an arc $\alpha_i \to \alpha_j$ in D if and only if there is an arc $\alpha_i \to \alpha_j$ in D^+. Thus, $D = D^+$ and so $B = B^+$. The h-graph G is strongly self-dual. $\square$

4. Constructions of strongly self-dual graphs

4.1. Pendant vertices and edges

In an undirected graph, a *pendant vertex* is one that has degree equal to 1 and a *pendant edge* is an edge incident to a pendant vertex. Note that in the canonical representation of an h-graph (see Proposition 1.3), the vertices u_1 and v_m both have degree equal to 1. Thus, every h-graph possesses at least two pendant vertices – one in each bipartite class.

Lemma 4.1. *Let G be an h-graph and let $B = B_G$. Then,*

$$B' = \begin{bmatrix} 1 & 0 \\ v & B \end{bmatrix}$$

is signature-similar to its inverse if and only if there is a signature matrix S such that $SBS = B^{-1}$ and $BSv = \pm v$.

Proof. Suppose that the signature matrix

$$S' = \begin{bmatrix} -t & 0 \\ 0 & S \end{bmatrix}$$

satisfies $S'B'S' = (B')^{-1}$. Then, we have

$$\begin{bmatrix} 1 & 0 \\ -tSv & SBS \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -B^{-1}v & B^{-1} \end{bmatrix}.$$

Thus, $SBS = B^{-1}$ and $tSv = B^{-1}v = SBSv$ where $t = \pm 1$. Since $S^2 = I$, we have $BSv = tv = \pm v$. Conversely, if $SBS = B^{-1}$ and $BSv = tv$ where $t = \pm 1$, then the signature matrix

$$S' = \begin{bmatrix} -t & 0 \\ 0 & S \end{bmatrix}$$

conjugates B' to its inverse. $\square$

Corollary 4.2. *Let G' be a strongly self-dual h-graph, let $B' = B_{G'}$, and express*

$$B' = \begin{bmatrix} 1 & 0 \\ v & B \end{bmatrix}.$$

Then, $B = B_G$ where G is a strongly self-dual h-graph that is an induced subgraph of G'.

When G' and G satisfy the conditions of Corollary 4.2, we say that G' is constructed by *adding a pendant to G*. Evidently, every strongly self-dual h-graph can be constructed from a smaller strongly self-dual h-graph in this manner. Given a strongly self-dual G, can we form a larger h-graph by adding a pendant? If so, what can we say about such an h-graph?

We refer to the column vector that has its ith entry equal to 1 and every other entry equal to 0 as e_i (the order of e_i is determined by context).

Theorem 4.3. *Let G be a strongly self-dual h-graph; let $B = B_G$ and signature matrix S satisfy $SBS = B^{-1}$. Express the ith column of B as $e_i + c_i$. Then, $BSv = v$ if and only if v is a linear combination of the vectors*

$$\Lambda = \left\{ e_i + \frac{1}{2}c_i : s_i = 1 \right\}.$$

Proof. We aim to show that $\text{span}(\Lambda) = \ker(I - BS)$. Note that the vectors

$$v_i = e_i + \frac{1}{2}c_i$$

are linearly independent – the leading entry of v_i is contained in the ith row. Thus, the dimension of $\text{span}(\Lambda)$ is equal to the number of ones on the diagonal of S. As well, BS is a lower-triangular matrix and the diagonal of BS is equal to that of S. So the dimension of $\ker(BS - I)$ is less than or equal to the number of ones on the diagonal of S. Therefore, we need only show that $\text{span}(\Lambda) \subseteq \ker(I - BS)$.

Since $SBS = B^{-1}$, we have $B(SBS) = (BS)^2 = I$. The ith column of BS is simply the ith column of B multiplied by s_i, which we note is equal to $s_ie_i + s_ic_i$. Moreover, for any matrix X, Xe_i is the ith column of X. So,

$$\begin{aligned} e_i &= (BS)^2\, e_i \\ &= BS\,(s_ie_i + s_ic_i) \\ &= s_iBSe_i + s_iBSc_i \\ &= s_i\,(s_ie_i + s_ic_i) + s_iBSc_i \\ &= e_i + c_i + s_iBSc_i. \end{aligned}$$

Thus, for all i, $BSc_i = -s_ic_i$. Now, suppose that $s_i = 1$. Then,

$$\begin{aligned} BS\left(e_i + \tfrac{1}{2}c_i\right) &= BSe_i + \tfrac{1}{2}BSc_i \\ &= (s_ie_i + s_ic_i) + \tfrac{1}{2}\,(-s_ic_i) \\ &= e_i + c_i - \tfrac{1}{2}c_i \\ &= e_i + \tfrac{1}{2}c_i. \end{aligned}$$

Therefore, $\text{span}(\Lambda) \subseteq \ker(I - BS)$. $\square$

Corollary 4.4. *Let G be a strongly self-dual h-graph; let $B = B_G$ and signature matrix S satisfy $SBS = B^{-1}$. Express the ith column of B as $e_i + c_i$. Then, $BSv = -v$ if and only if v is a linear combination of the vectors*

$$\Lambda^- = \left\{ e_i + \frac{1}{2}c_i : s_i = -1 \right\}.$$

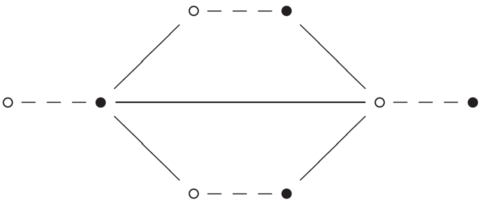

Fig. 1. H_1.

Corollary 4.5. *Let G be a strongly self-dual h-graph; let $B = B_G$ and signature matrix S satisfy $SBS = B^{-1}$. Express the ith column of B as $e_i + c_i$. Then, if v is a linear combination of the vectors*

$$K = \{c_i : s_i = -1\},$$

we have $BSv = v$.

Proof. We saw in the proof of Theorem 4.3 that for all i, $BSc_i = -s_i c_i$. Thus, if $s_i = -1$, we have $BSc_i = c_i$. By Lemma 4.1, the statement holds. □

Remark. If Λ and K are as in Theorem 4.3 and Corollary 4.5, Corollary 4.5 tells us that $\text{span}(K) \subseteq \text{span}(\Lambda)$. However, it can be the case that $\text{span}(K) \neq \text{span}(\Lambda)$

In diagrams of h-graphs, we use circles and dots to represent the bipartite classes; as well, we use dashed lines to represent the edges in the unique perfect matching.

Example 4.6. The h-graph H_1 (Fig. 1) is strongly self-dual.
We see this by noting that

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

and

$$B^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 1 \end{bmatrix} = SBS,$$

where $S = \text{diag}(-1, 1, 1, -1)$. We apply Corollary 4.5 with

$$v = c_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

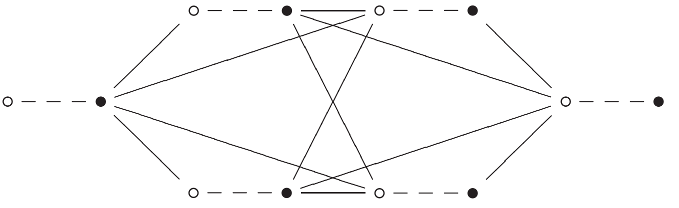

Fig. 2. H_2.

and obtain

$$B' = \begin{bmatrix} 1 & 0 \\ v & B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

The signature matrix $S' = \mathrm{diag}(-1, -1, 1, 1, -1)$ conjugates B' to its inverse. We then apply Theorem 4.3 with

$$v' = \left(e_1 + \frac{1}{2}c_1\right) + \left(e_2 + \frac{1}{2}c_2\right) - \left(e_5 + \frac{1}{2}c_5\right) = \begin{bmatrix} 1 \\ 0 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

obtaining

$$B'' = \begin{bmatrix} 1 & 0 \\ v' & B' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

with associated signature matrix $S'' = \mathrm{diag}(1, -1, -1, 1, 1, -1)$. Thus, H_2 (Fig. 2) is strongly self-dual.

5. Eigenvalues of self-dual *h*-graphs

The eigenvalues of a graph are the eigenvalues of its adjacency matrix. An algebraic integer is a real number that is the root of a monic polynomial with integer coefficients. The eigenvalues of a matrix with integral entries are algebraic integers and thus the eigenvalues of a graph are algebraic integers.

We consider the eigenvalues of self-dual h-graphs, with emphasis on the question of whether ± 1 are eigenvalues.

Theorem 5.1. *Let G be a self-dual h-graph and let λ be an eigenvalue of G. Then,*

$$-\lambda,\ \frac{1}{\lambda}\quad \textit{and}\quad -\frac{1}{\lambda}$$

are all eigenvalues of G, each with the same multiplicity as λ.

Proof. Let λ be an eigenvalue of G. It is well-known that since G is bipartite, $-\lambda$ is also an eigenvalue with the same multiplicity as λ. Moreover, the adjacency matrix of G is similar to its inverse, implying that $1/\lambda$ is an eigenvalue, again with the same multiplicity as λ. We then apply this second conclusion to $-\lambda$ to show that $-1/\lambda$ is also an eigenvalue with that same multiplicity. □

Let G be a strongly self-dual h-graph. Theorem 5.1 implies that whenever λ is an eigenvalue of G, both λ and $1/\lambda$ are algebraic integers. It is straightforward to show that this is, in fact, true of the eigenvalues of any graph whose adjacency matrix has determinant equal to ± 1. However, in the case of G an h-graph, we are able to determine that not only is $1/\lambda$ also an algebraic integer, it is also an eigenvalue of G.

Corollary 5.2. *Let G be a self-dual h-graph. Then, every eigenvalue of G is either ± 1 or irrational.*

Proof. Every eigenvalue of any graph is an algebraic integer; moreover, the only rational algebraic integers are the integers. Thus, if λ is a rational eigenvalue of G, then λ and $1/\lambda$ are both integers. □

Theorem 5.3. *Let G be a self-dual graph on $n = 2m$ vertices. Then, the multiplicity of ± 1 as eigenvalues is congruent to m modulo 2.*

Proof. Let the multiplicity of 1 as an eigenvalue of G be q (possibly $q = 0$). The multiplicity of -1 as an eigenvalue is then q, as well. Via Theorem 5.1, we see that the total of the multiplicities of the eigenvalues of G that are not equal to ± 1 is a multiple of 4, say this number is $4r$. Thus, $2q + 4r = 2m$, implying that $q = m - 2r$. □

Corollary 5.4. *Let G be a self-dual h-graph on $n = 2m$ vertices where m is odd. Then, ± 1 are eigenvalues of G.*

Corollary 5.4 provides a rather weak lower bound on the multiplicity of ± 1 as eigenvalues of a self-dual graph. We show that, in fact, a stronger lower bound exists in the case that G is strongly self-dual.

Theorem 5.5. *Let G be strongly self-dual on $2m$ vertices with $B_G = B$ and let S be a signature matrix such that $SBS = B^{-1}$. Let k^+ and k^- be the number of 1s and -1s on the diagonal of S, respectively. Then, the multiplicity of ± 1 as eigenvalues of the adjacency matrix of G is greater than or equal to the difference $|k^+ - k^-|$.*

Proof. Note that $SBS = B^{-1}$ if and only if $(-S)B(-S) = B^{-1}$; so, we will assume that $k^+ \geqslant k^-$ and we will show that the multiplicity of ± 1 as eigenvalues is at least $k^+ - k^- = 2k^+ - m$ (since $k^+ + k^- = m$).

Since $SBS = B^{-1}$, we have $(BS)^2 = I$. Therefore, the dimension of both

$$R = \ker\,(I - BS) \quad \text{and} \quad L = \ker\left(I - (BS)^T\right)$$

is k^+. (The geometric and algebraic multiplicities of 1 as an eigenvalue of BS are both equal to 1.) Now, $\dim(R + L) \leqslant m$ and

$$\dim(R + L) = \dim(R) + \dim(L) - \dim(R \cap L) = 2k^+ - \dim(R \cap L).$$

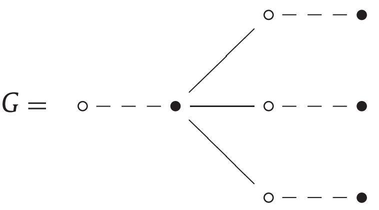

Fig. 3. *H*.

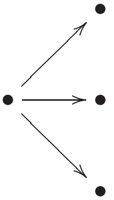

Fig. 4. D_H.

Together, these imply that

$$\dim(R \cap L) = 2k^+ - \dim(R + L) \geqslant 2k^+ - m.$$

Let $x \in R \cap L$. Then,

$$BSx = x = (BS)^T x = SB^T x.$$

Thus, since $SBS = B^{-1}$, $B^{-1}x = B^T x$. Then we note that

$$\begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix} \begin{bmatrix} x \\ B^{-1}x \end{bmatrix} = \begin{bmatrix} BB^{-1}x \\ B^T x \end{bmatrix} = \begin{bmatrix} x \\ B^{-1}x \end{bmatrix}$$

So, the dimension of the eigenspace of $M(G)$ associated with $\lambda = 1$ is at least $2k^+ - m$. $\square$

Note that Corollary 5.4 can be taken to be a corollary to Theorem 5.5; we have $k^+ + k^- = m$, so if m is odd then $|k^+ - k^-|$ is odd and thus greater than or equal to one.

Example 5.6. The h-graph H (Fig. 3) is the corona of a tree; thus, by Theorem 2.9, it is a strongly self-dual h-graph. It has order 8; so, Theorem 5.3 does not tell us whether or not it has ± 1 as eigenvalues. However, an examination of D_H (Fig. 4)shows that it, in fact, has ± 1 as eigenvalues.
By Corollary 2.4, the associated signature matrix has three entries of one sign and one entry of the opposite sign. Thus, the multiplicity of ± 1 as eigenvalues is at least $|k^+ - k^-| = 2$.

5.1. Eigenvectors and eigenvalues of coronas

The following Lemma is a straightforward application of the definition of the corona of a graph.

Lemma 5.7. *Let X be bipartite with adjacency matrix*

$$A = \begin{bmatrix} 0 & C \\ C^T & 0 \end{bmatrix},$$

where C is $m \times n$. Then the adjacency matrix of $G = C(X)$, the corona of X, can be expressed as

$$M = \begin{bmatrix} 0 & 0 & I_n & 0 \\ 0 & 0 & C & I_m \\ I_n & C^T & 0 & 0 \\ 0 & I_m & 0 & 0 \end{bmatrix}.$$

Theorem 5.8. *Let X be a bipartite graph and let $G = C(X)$ be the corona of X. Let λ be an eigenvalue of X with multiplicity k; then, the two distinct roots of $z^2 - \lambda z - 1$ are eigenvalues of G, each with multiplicity k.*

Proof. Express the adjacency matrices of X and G as in Lemma 5.7. Let v be an eigenvector associated with λ. We express

$$v = \begin{bmatrix} x \\ y \end{bmatrix},$$

where $Cy = \lambda x$ and $C^T x = \lambda y$. Let μ be a root of $z^2 - \lambda z - 1$; note that this implies $\mu^2 = \lambda\mu + 1$. We then produce an eigenvector associated with μ:

$$\begin{bmatrix} 0 & 0 & I_n & 0 \\ 0 & 0 & C & I_m \\ I_n & C^T & 0 & 0 \\ 0 & I_m & 0 & 0 \end{bmatrix} \begin{bmatrix} y \\ \mu x \\ \mu y \\ x \end{bmatrix} = \begin{bmatrix} \mu y \\ \mu C y + x \\ y + \mu C^T x \\ \mu x \end{bmatrix} = \begin{bmatrix} \mu y \\ \mu\lambda x + x \\ y + \mu\lambda y \\ \mu x \end{bmatrix} = \begin{bmatrix} \mu y \\ \mu^2 x \\ \mu^2 y \\ \mu x \end{bmatrix}.$$

Linearly independent eigenvectors for λ produce linearly independent eigenvectors for μ. Moreover, the fact that G has order twice that of X implies that we have accounted for all the eigenvalues of G. $\square$

Let $\lambda \neq 0$ be an eigenvalue of a bipartite graph X with multiplicity k; again, $-\lambda$ is an eigenvalue with equal multiplicity. So, the roots of $z^2 - \lambda z - 1$ and $z^2 + \lambda z - 1$ are eigenvalues of $G = C(X)$ with multiplicity k. Let μ be a root of $z^2 - \lambda z - 1$, then the other root is $-1/\mu$. Moreover, the roots of $z^2 + \lambda z - 1$ are then $-\mu$ and $1/\mu$. Thus, the eigenvalues of a corona that are not derived from eigenvalue 0 appear in the required groups of 4 as in Theorem 5.1.

Corollary 5.9. *Let $G = C(X)$ be the corona of a bipartite graph. The multiplicity of ± 1 as eigenvalues of G is equal to the multiplicity of 0 as an eigenvalue of X.*

Proof. The only polynomial of the form $z^2 - \lambda z - 1$ that has 1 as a root is $z^2 - 1$. $\square$

5.2. Eigenvalues of strongly self-dual subgraphs

We are interested in the eigenvalues of strongly self-dual pairs G and G', where G' is formed by adding a pendant to G. Recall that if G and G' are h-graphs such that $B = B_G$ and $B' = B_{G'}$ satisfy

$$B' = \begin{bmatrix} 1 & 0 \\ v & B \end{bmatrix},$$

then we say that G' is formed by adding a pendant to G.

Theorem 5.10. *Let G and be an h-graph and construct G' by adding a pendant to G. Let λ be an algebraic integer and let the multiplicities of λ as an eigenvalue of G and G' be s and s', respectively. Then, s and s' differ by at most* 1.

Proof. Note that either of s or s' might be equal to zero. We will proceed by showing that $s' \geqslant s - 1$ and that $s \geqslant s' - 1$. Clearly,

$$\begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}$$

if and only if $By = \lambda x$ and $B^T x = \lambda y$. In a similar manner,

$$w' = \begin{bmatrix} \alpha & x^T & \beta & y^T \end{bmatrix}^T$$

is an eigenvector for $M(G')$ associated with λ if and only if

$$\begin{aligned} \beta &= \lambda\alpha, \\ \beta v + By &= \lambda x, \\ \alpha + v^T x &= \lambda\beta, \quad \text{and} \\ B^T x &= \lambda y. \end{aligned}$$

If $s = 0$ or 1, then $s' \geqslant 0 \geqslant s - 1$; so, we assume that s is at least 2. Let the vectors

$$\left\{ w_i = \begin{bmatrix} x_i^T & y_i^T \end{bmatrix}^T : 1 \leqslant i \leqslant s \right\}$$

be a linearly independent collection of eigenvectors of $M(G)$ associated with λ. For each i, if $v^T x_i = 0$ then we can lift w_i to

$$w_i' = \begin{bmatrix} 0 & x_i^T & 0 & y_i^T \end{bmatrix}^T,$$

which is an eigenvector of $M(G')$ associated with λ. (It is easy to see that if $\alpha = \beta = v^T x = 0, By = \lambda x$ and $B^T x = \lambda y$ then the four conditions above are all satisfied.) Moreover, if there are multiple such x_is, then the obtained w_i's are linearly independent. Now, if every $v^T x_i = 0$, this implies that $s' \geqslant s$. If not, assume that $v^T x_s \neq 0$. Then, the collection

$$\left\{ \begin{bmatrix} x_i \\ y_i \end{bmatrix} - \frac{v^T x_i}{v^T x_s} \begin{bmatrix} x_s \\ y_s \end{bmatrix} : 1 \leqslant i \leqslant s - 1 \right\}$$

can be lifted to a collection of $s - 1$ linearly independent eigenvectors of $M(G')$ associated with λ. Thus, in any case, $s' \geqslant s - 1$.

As before, if $s' = 0$ or 1 we have $s \geqslant s' - 1$. So, suppose that $s' \geqslant 2$ and let

$$\left\{ w_i' = \begin{bmatrix} \alpha_i & x_i^T & \beta_i & y_i^T \end{bmatrix}^T : 1 \leqslant i \leqslant s' \right\}$$

be a linearly independent collection of eigenvectors of $M(G')$ associated with λ. For each i, if $\alpha_i = 0$, then $\beta_i = \lambda\alpha_i = 0$ and we then have $By_i = \lambda x_i$ and $B^T x_i = \lambda y_i$. Thus, each such w_i' can be projected

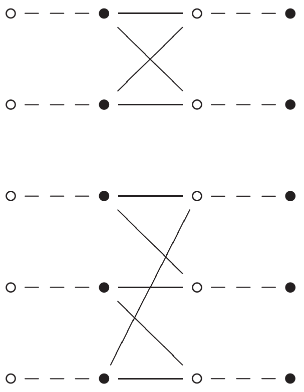

Fig. 5. G_0.

onto an eigenvector for $M(G)$,

$$w_i = \left[\, x_i^T \;\; y_i^T \,\right]^T.$$

If every $\alpha_i = 0$ we then have $s \geqslant s'$. Otherwise, we assume that $\alpha_{s'} \neq 0$; then

$$\left\{ w_i' = \left[\, \alpha_i \;\; x_i^T \;\; \beta_i \;\; y_i^T \,\right]^T - \frac{\alpha_i}{\alpha_{s'}} \left[\, \alpha_{s'} \;\; x_{s'}^T \;\; \beta_i \;\; y_i^T \,\right]^T : 1 \leqslant i \leqslant s' - 1 \right\}$$

can be projected onto a linearly independent collection of $s' - 1$ eigenvectors of $M(G)$ associated with λ. Therefore, in any case, we also have $s \geqslant s' - 1$. □

Corollary 5.11. *Let G and G' be self-dual h-graphs and suppose that G' is formed by adding a pendant to G. Then, the multiplicities of ± 1 as eigenvalues of G and G' differ by exactly one.*

Proof. Let G have order $2m$; then G' has order $2(m+1)$. Let s and s' be the respective multiplicities of 1 as an eigenvalue of the two graphs. By Theorem 5.3, one of s or s' is odd and the other is even and, by Theorem 5.10, we have $|s' - s| \leqslant 1$. Thus, $|s' - s| = 1$. □

Example 5.12. The following series of strongly self-dual h-graphs are an interesting example of some of the propositions we have seen so far.
The h-graph G_0 (Fig. 5) is the corona of a union of bipartite cycles and so is strongly self-dual. We construct G_1 (Fig. 6) by adding a pendant to G_0 and then construct G_2 (Fig. 7) by adding a pendant to G_1. It can be shown that both additional pendants satisfy Theorem 4.3; G_1 and G_2 are strongly self-dual, as well. The characteristic polynomials of these graphs are:

$$\begin{aligned}
p_0(z) &= \left(z^2-1\right)^2 \left(z^2-z-1\right)^2 \left(z^2+z-1\right)^2 \\
&\quad \times \left(z^2-2z-1\right)^2 \left(z^2+2z-1\right)^2 \\
p_1(z) &= \left(z^2-1\right)^3 \left(z^2-z-1\right)^2 \left(z^2+z-1\right)^2 \\
&\quad \times \left(z^2-2z-1\right) \left(z^2+2z-1\right) \left(z^4-16z^2+1\right) \\
p_2(z) &= \left(z^2-1\right)^2 \left(z^2-z-1\right)^2 \left(z^2+z-1\right)^2 \\
&\quad \times \left(z^2-2z-1\right) \left(z^2+2z-1\right) \left(z^8-28z^6+154z^4-28z^2+1\right)
\end{aligned}$$

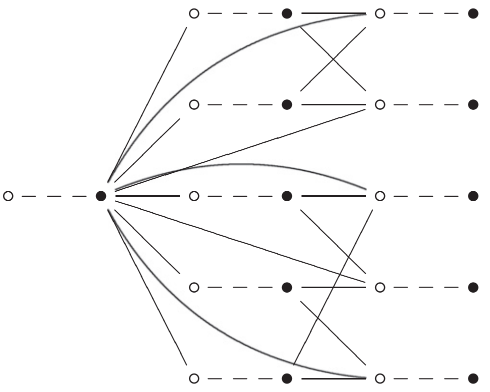

Fig. 6. G_1.

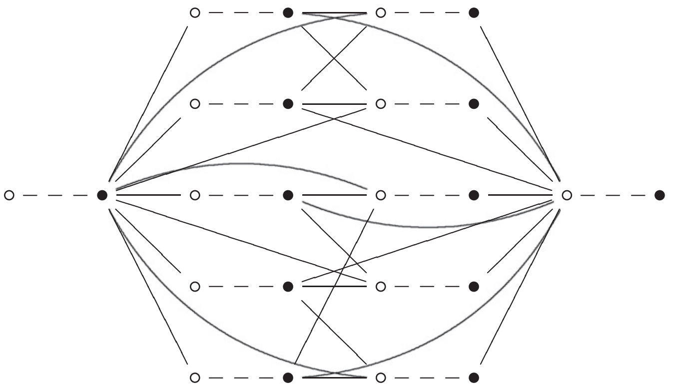

Fig. 7. G_2.

We can calculate $p_0(z)$ using Theorem 5.8 and the well-known spectrum of a cycle; the other polynomials are the result of an investigation into interval h-graphs found in [10], which we do not reproduce here. The roots of $(z^2 \pm z - 1)$ are eigenvalues of all three graphs with identical multiplicities. The addition of the first pendant increases the multiplicities of ± 1 by one, decreases the multiplicities of the roots of $(z^2 \pm 2z - 1)$ by one and introduces four new eigenvalues; the second additional pendant decreases the multiplicities of several eigenvalues by one and introduces eight new eigenvalues. Thus, we see all three possible behaviors predicted by Theorem 5.10. Further, it is straightforward to show that G_0 and G_2 have $|k^+ - k^-| = 0$ and G_1 has $|k^+ - k^-| = 1$; thus, these graphs have ± 1 as eigenvalues with multiplicities strictly higher than the lower bound given by Theorem 5.5.

Self-dual and strongly self-dual h-graphs are incredibly interesting classes of invertible graphs. They have a rich structure and display an intimate relationship between their spectral and combinatoric properties.

Several interesting open problems remain with regards to this relationship. It can be shown that in the above three examples (G_0, G_1 and G_2), the eigenspaces associated with the eigenvalues ± 1 have bases consisting of $(0, \pm 1)$-vectors; classifying exactly when this holds for the general h-graph appears to be a challenging problem. Via Corollary 5.11, we see that if we add a pendant to a strongly self-dual h-graph in such a way that the obtained graph is also strongly self-dual, then the multiplicity of ± 1 as eigenvalues has changed by exactly one. Classifying exactly when this change is an increase versus a decrease seems very related to similar problems involving eigenvalue interlacing and remains unsolved.

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